

# ON THE THREE-DIMENSIONAL TEMPORAL SPECTRUM OF STRETCHED VORTICES

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The three-dimensional stability problem of a stretched stationary vortex is addressed in this letter. More specifically, we prove that the discrete part of the temporal spectrum is only associated with two-dimensional perturbations.

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## I. INTRODUCTION

Numerical simulations [1–3] as well as real experiments [4] indicate that vorticity in turbulent flows concentrates in localized regions such as filaments which are fairly well described by stretched vortices such as the celebrated Burgers vortex solution [5] or Moffatt, Kida & Ohkitani's asymptotic solution [6]. If one agrees that these local structures are important dynamical objects of the global turbulent field, their temporal stability with respect to generic perturbations should be addressed. So far, this problem has only been studied for Burgers' vortex and, in that case, for purely two-dimensional perturbations. Robinson & Saffman [7] provided an analytical solution for low-Reynolds numbers and their result were later numerically extended by Prochazka & Pullin [8] up to Reynolds numbers  $Re = 10^4$ . These papers on Burgers vortex indicate that the temporal spectrum associated with two-dimensional perturbations is discrete and corresponds to damped modes. On the contrary, the general stability problem of stretched vortices has not been tackled yet. Except for the 2D stability analysis of axisymmetric Burgers vortex, it does not reduce to a classical eigenvalue problem with a single ODE to solve. Indeed, infinitesimal 3D perturbations are affected by the presence of stretching along the vortex axis which precludes the reduction of the problem by Fourier analysis.

In this paper, we prove that the discrete part of the temporal spectrum is only associated with two-dimensional perturbations. In section II, the stability problem is introduced and particular time-dependent solutions are exhibited. Their existence imposes conditions on the 3D temporal mode structure. In section III, these conditions are shown to be consistent only for modes independent of the vortex axis coordinate.

## II. MODIFIED FOURIER DECOMPOSITION

Let us consider a stationary velocity field  $\mathbf{U}_0 = (U_0, V_0, W_0)$  of the form

$$U_0 = \frac{\partial \phi}{\partial x}(x, y) + U_v(x, y), \quad (2.1a)$$

$$V_0 = \frac{\partial \phi}{\partial y}(x, y) + V_v(x, y), \quad (2.1b)$$

$$W_0 = \gamma z, \quad (2.1c)$$

where  $(U_v, V_v)$  and  $(\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y), \gamma z)$  respectively stand for a localized rotational field and a global velocity field satisfying  $\nabla^2 \phi = -\gamma$ . Such an expression represents a stationary stretched vortex aligned with the  $z$ -axis and subjected to a global strain field. In the sequel, the strain rate  $\gamma$  along the vortex axis is assumed to be positive.

The structure of such a solution is governed by the balance between stretching due to the global strain field and viscous diffusion. In particular, the core size should scale as  $\sqrt{\nu/\gamma}$  where  $\nu$  is the kinematic viscosity. In the simplest case of an axisymmetric strain  $(\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y), \gamma z) = (-\gamma x/2, -\gamma y/2, \gamma z)$ , one recovers the Burgers

solution. Other examples are Robinson & Saffman [7] and Moffatt, Kida & Ohkitani [6] solutions which correspond to the non-axisymmetric case at small and large Reynolds numbers respectively. In the subsequent analysis, expression (2.1a-c) is considered as the basic flow where  $(U_v, V_v)$  and  $\phi$  are not specified.

The dynamics of pressure and velocity infinitesimal perturbations  $(\mathbf{u}, p)$  around (2.1a-c) is described by the linear system :

$$\partial_t \mathbf{u} + \mathbf{U}_0 \cdot \nabla \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{U}_0 = -\nabla p + \nu \nabla^2 \mathbf{u} , \quad (2.2a)$$

$$\nabla \cdot \mathbf{u} = 0 . \quad (2.2b)$$

The above system being homogeneous with respect to time, one might look at the temporal spectrum of such system. Modes belonging to the discrete part of this spectrum read:

$$(\mathbf{u}_\omega, p_\omega) = (\mathbf{v}_\omega(x, y, z), q_\omega(x, y, z)) e^{-i\omega t} . \quad (2.3)$$

Inserting (2.3) into (2.2a-b), one obtains equations for  $\mathbf{v}_\omega(x, y, z)$  and  $q_\omega(x, y, z)$  which are non-separable with respect to any spatial variable. In such a case, the use of standard Fourier analysis does not simplify any further the problem. However, the  $z$ -dependence in equation (2.2a) only appears through the uniform strain along the  $z$ -axis i.e. through the term  $\gamma z \partial_z$ . Such a simple dependence allows to search for time-dependent solutions which are different from (2.3). These are “generalized Fourier modes” in the  $z$  direction with a time-dependent wavenumber :

$$(\mathbf{u}, p) = (\check{\mathbf{u}}(x, y, t, k_0), \check{p}(x, y, t, k_0)) e^{ik(t)z} e^{-\nu \int_0^t (k(s))^2 ds} , \quad (2.4)$$

where the initial wavenumber condition  $k_0 = k(0)$  is a free parameter. Indeed, as soon as the time evolution of  $k(t)$  is appropriately chosen, more precisely if  $k(t) = k_0 e^{-\gamma t}$ , the non-homogeneous term in  $z$  is removed in (2.2a). The system (2.2a,b) is then reduced to a couple of equations homogeneous in  $z$ , which describes the time evolution of the modified Fourier modes components  $\check{u}_x$  and  $\check{u}_y$  :

$$(\mathcal{L} + \partial_x U_0) \check{u}_x + \partial_y U_0 \check{u}_y = \frac{e^{2\gamma t}}{k_0^2} \frac{\partial}{\partial x} (\mathcal{L} + 2\gamma) \left( \frac{\partial \check{u}_x}{\partial x} + \frac{\partial \check{u}_y}{\partial y} \right) , \quad (2.5a)$$

$$(\mathcal{L} + \partial_y V_0) \check{u}_y + \partial_x V_0 \check{u}_x = \frac{e^{2\gamma t}}{k_0^2} \frac{\partial}{\partial y} (\mathcal{L} + 2\gamma) \left( \frac{\partial \check{u}_x}{\partial x} + \frac{\partial \check{u}_y}{\partial y} \right) , \quad (2.5b)$$

where

$$\mathcal{L} = \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x} + V_0 \frac{\partial}{\partial y} - \nu \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) . \quad (2.6)$$

Finally, note that  $\check{u}_z$  is given by the components  $\check{u}_x$  and  $\check{u}_y$  through the continuity equation.

The temporal mode (2.3) may be decomposed upon a basis of such modified Fourier modes (2.4). Let us expand the spatial part  $(\mathbf{v}_\omega, q_\omega)$  of (2.3) in the usual Fourier modes along the  $z$  direction :

$$(\mathbf{v}_\omega, q_\omega) = \int_{-\infty}^{+\infty} (\mathbf{v}(x, y, k_0), q(x, y, k_0)) e^{ik_0 z} dk_0 . \quad (2.7)$$

This expansion can be viewed as a superposition at time  $t = 0$  of generalized Fourier modes provided the following initial condition is satisfied :

$$(\check{\mathbf{u}}(x, y, 0, k_0), \check{p}(x, y, 0, k_0)) = (\mathbf{v}(x, y, k_0), p(x, y, k_0)) . \quad (2.8)$$

According to (2.4), each generalized Fourier mode evolves independently: an alternative expression for the temporal mode (2.3) is thus provided for all  $t$

$$(\mathbf{u}_\omega, p_\omega) = \int_{-\infty}^{+\infty} (\check{\mathbf{u}}(x, y, t, k_0), \check{p}(x, y, t, k_0)) e^{ik_0 e^{-\gamma t} z} e^{-\nu k_0^2 (1 - e^{-2\gamma t})/2\gamma} dk_0 . \quad (2.9)$$

The above expression is consistent with (2.3) if the following equality holds at any  $x$  and  $y$  locations, time  $t$  and wavenumber  $k_0$  :

$$(\check{\mathbf{u}}(x, y, t, k_0), \check{p}(x, y, t, k_0)) e^{-\nu k_0^2 (1 - e^{-2\gamma t})/2\gamma} = (\mathbf{v}(x, y, k_0 e^{-\gamma t}), q(x, y, k_0 e^{-\gamma t})) e^{-\gamma t} e^{-i\omega t} . \quad (2.10)$$

In the following section, equality (2.10) is shown to be valid only for  $k_0 = 0$ : discrete temporal modes are bound to be two-dimensional.

### III. THE $z$ DEPENDENCE OF A THREE-DIMENSIONAL TEMPORAL MODE

First a cut-off wavenumber  $k_c$  above which  $\mathbf{v}(x, y, k_0)$  and  $q(x, y, k_0)$  vanish, should exist. Indeed, assume that such a cut-off does not appear, large wavenumbers are then present in the spatial spectrum of (2.3). It is thus possible to take the simultaneous limits  $k_0 t$  large and  $t$  small in (2.10). The right-hand side of this equation becomes

$$(\mathbf{v}(x, y, k_0 e^{-\gamma t}), q(x, y, k_0 e^{-\gamma t})) e^{-\gamma t} e^{-i\omega t} \sim (\mathbf{v}(x, y, k_0), q(x, y, k_0)) . \quad (3.1)$$

In order to estimate the left-hand side of (2.10), the behavior of  $\check{\mathbf{u}}(x, y, t, k_0), \check{p}(x, y, t, k_0)$  is evaluated using equations (2.5a,b). Two cases are to be considered according to the characteristic spatial variations of  $\check{u}_x$  and  $\check{u}_y$  in the  $x$  and  $y$  directions. When, for large axial wavenumber  $k_0$ , these components evolve over spatial scales independent of  $k_0$ , the right-hand side of (2.5a,b) can be neglected and the leading order time evolution is independent on  $k_0$ . This means that, for  $k_0 t$  large and  $t$  small, the left-hand side of (2.10) reads

$$(\check{\mathbf{u}}(x, y, t, k_0), \check{p}(x, y, t, k_0)) e^{-\nu k_0^2 (1 - e^{-2\gamma t})/2\gamma} \sim (\check{\mathbf{u}}_\infty(x, y, 0), \check{p}_\infty(x, y, 0)) e^{-\nu k_0^2 t} . \quad (3.2)$$

On the contrary, when  $\check{u}_x, \check{u}_y$  evolve over spatial scales comparable to  $1/k_0$ , the correct expression is found when introducing in (2.5a,b) new fast variables

$$\bar{x} = k_0 x ; \bar{y} = k_0 y ; \bar{t} = k_0^2 t, \quad (3.3)$$

in addition to  $x$  and  $y$ , and thereafter expanding the solution with respect to  $1/k_0$ . At leading order, homogeneous equations are obtained :

$$\frac{\partial \check{u}_x}{\partial \bar{t}} - \nu \left( \frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} \right) \check{u}_x = e^{2\gamma t} \left[ \frac{\partial}{\partial \bar{t}} - \nu \left( \frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} \right) \right] \frac{\partial}{\partial \bar{x}} \left( \frac{\partial \check{u}_x}{\partial \bar{x}} + \frac{\partial \check{u}_y}{\partial \bar{y}} \right) , \quad (3.4a)$$

$$\frac{\partial \check{u}_x}{\partial \bar{t}} - \nu \left( \frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} \right) \check{u}_y = e^{2\gamma t} \left[ \frac{\partial}{\partial \bar{t}} - \nu \left( \frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} \right) \right] \frac{\partial}{\partial \bar{x}} \left( \frac{\partial \check{u}_x}{\partial \bar{x}} + \frac{\partial \check{u}_y}{\partial \bar{y}} \right) . \quad (3.4b)$$

For  $t$  small, the term  $e^{2\gamma t}$  may be taken equal to 1 and the general solution of (3.4a) be written in the fast variables  $\bar{x}$  and  $\bar{y}$  via the usual Fourier decomposition

$$(\check{u}_x, \check{u}_y) = \int (\check{u}_x^{(0)}(k_x, k_y, x, y), \check{u}_y^{(0)}(k_x, k_y, x, y)) e^{-\nu(k_x^2 + k_y^2)\bar{t}} e^{ik_x \bar{x} + ik_y \bar{y}} dk_x dk_y . \quad (3.5)$$

For fixed  $x$  and  $y$  and  $k_0 t \rightarrow +\infty$ , the above expression (3.5) is evaluated by the steepest descent method

$$(\check{u}_x, \check{u}_y) \sim \frac{4\pi}{\nu k_0^2 t} (\check{u}_x^{(0)}(0, 0, x, y), \check{u}_y^{(0)}(0, 0, x, y)) . \quad (3.6)$$

Similar behaviors can be obtained for  $\check{u}_z$  and  $\check{p}$ . For this case, the right-hand side of (2.10) now reads

$$(\check{\mathbf{u}}(x, y, t, k_0), \check{p}(x, y, t, k_0)) e^{-\nu k_0^2 (1 - e^{-2\gamma t})/2\gamma} \sim (\check{\mathbf{u}}^{(0)}(0, 0, x, y), \check{p}^{(0)}(0, 0, x, y)) \frac{4\pi}{\nu k_0^2 t} e^{-\nu k_0^2 t} . \quad (3.7)$$

Introducing into equation (2.10), estimates (3.1) and (3.2) or (3.7) leads to an inconsistency for large wavenumbers  $k_0$ . This contradiction implies that the spatial spectrum of (2.3) vanishes for sufficiently large wavenumbers : there hence exists a cut-off wavenumber  $k_c$  such that  $(\mathbf{v}(x, y, k_0), q(x, y, k_0)) = 0$  for  $k_0 > k_c$ .

Consider now a wavenumber  $0 < k_1 < k_c$  such that  $(\mathbf{v}(x, y, k_1), q(x, y, k_1)) \neq (\mathbf{0}, 0)$  and a time  $t_1$  such that  $k_0 = k_1 e^{\gamma t_1} > k_c$ . Equation (2.10) then implies that  $(\check{\mathbf{u}}(x, y, t_1, k_0), \check{p}(x, y, t_1, k_0)) \neq 0$  which leads again to a contradiction. Indeed, the initial condition  $(\check{\mathbf{u}}(x, y, 0, k_0), \check{p}(x, y, 0, k_0))$  is equal to zero since  $k_0 > k_c$ , the quantity  $(\check{\mathbf{u}}(x, y, t, k_0), \check{p}(x, y, t, k_0))$  thus remains zero for all  $t > 0$  since it is governed by equation (2.5a,b). This imposes that  $(\mathbf{v}(x, y, k_1), q(x, y, k_1)) = (\mathbf{0}, 0)$  for all  $k_1 \neq 0$ . Discrete temporal modes are then independent upon  $z$  i.e. purely two-dimensional ones. For the axisymmetric Burgers vortex, those have been computed by Robinson & Saffman [7] and Prochazka and Pullin [8]. These authors showed that these modes are damped.

Needless to say, the general stability problem for Burgers vortex or any stretched vortex, cannot be solved only looking at the discrete part of the temporal spectrum. In particular, the analysis of the continuous spectrum need to be considered. However, the process of “bidimensionalization” displayed in (2.4) can be used again in that case.

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